**Mathematics and Economics: detrimental influence or fruitful complementarity**

**Matematicas y economia, influencia prejudicial o complementariedad fructifera**

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It is a great honor to stand today in front of you and to give this talk on Mathematics and Economics: detrimental influence or fruitful complementarity after having been awarded this Honoris Causa Doctorate from your university.

The defence of my PhD dissertation took place in June 1988, 30 years ago. I am not sure that I was less moved than today. But I am convinced that this day will remain in my memory forever.

I would like first to express my deep thanks to the Rector Carlos Greco, to the Escuela de Economia y Negocios and its Dean Marcelo Paz that has proposed my name for this distinction and to the Consejo Superior de la UNSAM, which has awarded me this honoris causa doctorate and to the whole academic community.

**FRAME**

I wish to dedicate this presentation to the memory of Gérard Debreu, the Economic Nobel prize winner 1983. He was the president of my PhD committee in 1988 at the Université Paris 1 Panthéon-Sorbonne, he spent the last years of his life in Paris, he came in 1998, 20 years ago, with a Chair Blaise Pascal from the Ecole normale supérieure and I have now the honour to occupy his office at the Maison des sciences économiques.

I have also a deep scientific debt toward Bernard Cornet, who is now Professor emeritus at the Université Paris 1 and professor of economics at the University of Kansas. He was my PhD advisor, we have completed a lot of work jointly and he was the first in 1984, who called me to prepare a PhD when I was thinking of starting a job in the economic state administration.

This tradition of the Honoris Causa Doctorate is a major symbol of the fact that science has no border and that international exchanges and recognitions of major scientific contributions is not a matter of national preferences or choices but based on shared worldwide criterion of excellence and peer evaluation.

International cooperation and intellectual exchanges is the best mean to share knowledge, to progress altogether toward an ever growing understanding of the world, but also to share an ethical though on the social impact of scientific progress,of its implementation, and on the fair distribution of the social benefits.

I started 20 years ago to develop the international relations at the level of the department of mathematics and computer science in my university to improve the employability of our graduates and to offer them a larger range of possibilities to continue in PhD programs. That time was also the first years of the Erasmus exchange program in Europe.

14 years later, I received a four year mandate as vice-president of my university in charge of international relations. This field experience had a strong impact on my understanding of the best practices in the academic community. I am strongly convinced that we need to increase the students exchanges at the early stage of the higher education. These early exchanges provide a better language proficiency, an open mind and equip the students with an international network of classmates, which is a strong help for further studies abroad, the best way to reach the international excellence.

This exchanges must be implemented with the Erasmus spirit of the European Union, that is in a cooperative way between equal academic partners on the basis of transferable credits and recognition of the studies in the partner universities. This is at the opposite of the competitive behavior on the higher education market where each institution tries to attract the best students just to keep them in their own programs.

When I was a PhD student, I had the wonderful opportunity to get a scholarship to prepare my thesis at the Center of Operation Research and Econometrics of the Université catholique de Louvain. This center was and still is a highly internationalized laboratory, which offers me to meet many outstanding researchers in theoretical economics and econometrics. In particular, I met for the first time Gérard Debreu there. 30 years later, I am still in contact with several colleagues that I met during this stay in Louvain la Neuve and we are now building a joint master program with the European program Erasmus Mundus with the department of economics at UCL. So this is an evidence thay early stage exchanges have a long lasting effect.

Let me switch now to the main purpose of my presentation.

**FRAME**

A tradition of French engineers in economics:} dating back to the 19th century, L\'eon Walras, Antoine-Augustin Cournot, Nobel Prize Winners: Jean Tirole, Maurice Allais, \\

Roger Guesnerie, professor at the Coll\`ege de France, Edmond Malinvaud, Marcel Boiteux Thomas Piketty and many others

**FRAME**

A conference of Yvar Ekeland a well known mathematician in particular for its famous ``Ekeland variational principle", former president of the University Paris Dauphine, with major contributions in convex analysis, functional analysis and mathematical economics.

Conference June 23, 2000, Université de tous les savoirs

``The use of mathematical modeling in economics and more generally in social sciences, still shocks an audience who is used of the success of such modeling in experimental and natural sciences."

**FRAME**

Critics against mathematical modeling

- Can we describe a human being with equations? Where is the human freedom if economic phenomena are under the control of laws? \\

- A fundamental difference between the ``behaviour" of atoms or particules and human decision: the atoms cannot decide whether or not they will obey to the law of gravitation or the law of electromagnetism.\\

- The economic predictions are often wrong.\\

- A political debate: mathematical models are strongly oriented to support a liberal economic policy and the liberalization of the markets at the opposite of the social and fair policy taking into account the needs of the less advanced countries and populations.

**FRAME**

What happens in other sciences

- The compuptation of the age of the universe \\

with the speed of expansion, reverse dynamical equations where galaxies are considered as a dot!!

- The difficulties of understanding the transition between a micro structure where each element is considered in itself and the macro structure where we study the evolution of the system as a whole.\\

- The contradiction between several theories, like general relativity and quantum theory.\\

- The enigma of the dark matter.\\

- The meteorological previsions.\\

We can also compare with meteorology short term previsions versus long term, and mean temperature during a month versus the exact temperature in such a day.

**FRAME**

Why such a difference in social sciences?

A public discourse may have an effect on the behavior of the economic agents, so an effect on the final outcome of the social or economic process: this phenomenon is known as self fulfilling prophecies.

- Social sciences have a direct impact on our life, our well being, our health;\\

- We can act on our economic environment through dedicated policies, international agreements, individual choice, individual effort or investment;\\

- Everybody has a personal experience of economic and social environment and is invited to participate to the decision at least through the electoral processes.\\

**FRAME**

Limited power of mathematics to describe an economic system

A short story about mathematics and economics: future student saying that mathematics for economists are easier than the one for physics or mechanics. Clearly wrong since to describe an exchange, we need at least two agents and we need at least two commodities or one commodity and money. So, we are working in a four dimensional space whereas in mechanics we are only in a three dimensional space.

Economic phenomena concern a huge number of agents, of commodities, plus time and uncertainty;\\

So high dimensional problems with a lot of datas

Some concepts like preferences, risk aversion, are not easily measurable and representable by a numerical index;\\

Economic agents are not fully rational and not fully informed;\\

With the difficulties to measure to what extend are this lack of information or rationality

No real forces to drive the economic dynamics;\\

More and more datas but a problem of defining what is represented by these datas.

**FRAME**

Nevertheless a key tool for a better understanding

- Checking the internal consistency of some non formal reasoning and detection of counterintuitive mechanism;\\

If you increase the total wealth, does it increase the welfare of all economic agents?

- Defining a range of validity of some economic theories;\\

- Understanding the possibility of manipulation;\\

detecting frauds and cartels on markets or insider trading on financial markets or free-riding.

- Understanding what are the mechanisms where revealing the truth is optimal to obtain a fair outcome;\\

this is fundamental for example on insurance markets but also in voting procedures or in aggregation of individual preferences.

- Designing optimal taxation or optimal pricing in network industry, transportation, energy, telecommunication.\\

the example of auction theory for the allocation of radio frequencies for telecom operators

**FRAME**

What mathematics is not able to do

Note that we have a similar debate today, with the role of artificial intelligence and related algorithms in our daily life. The decision is taken by a machine, where is our freedom, can we control the outcome of the process?

- Providing a global theory to answer to all social and economic questions;\\

- To take a political decision when there is an arbitrage between several individual interests;\\

- Even for a well stylized fact, taken into account all interactions and all factors;\\

%for example, having a good description of probabilities in real events.

- Designing efficient algorithms to compute the exact solution of the economic equations.\\

% Most of the simple problems are NP Complete meaning that they are not computable in polynomial time with respect to datas.

**FRAME**

Keep a scientific attitude

Stay away from a fascination of numbers and equations. The use of mathematics in economics must be done within a framework defined by good practices and a clear understanding of the limit between the conclusions of the theories and their interpretation in a political context. The answer of Gérard Debreu to journalists: I never studied that topic.

- Understanding the limit of mathematical models as well as in the other sciences; \\

- Stay within the range of validity of a theory and avoid extrapolation;\\

- A mathematical model is only a support to clarify an economic question, it does not provide a solution;\\

mathematics for decision in Paris Dauphine University in the 70ties

- Checking the theories by an extensive use of econometric studies and experiments.\\

- Take into account the contribution of other social sciences like sociology, demography, geography, ....

\section{An example of a mathematical contribution to economics}

Bernard Cornet, Jean-Philippe M\'edecin, Matias Fuentes, Moncef Meddeb, Alexandrine Jamin, Antoine Mandel, Marc-Olivier Czarnecki, Bertrand Crettez, Lalaina Rakononindrainy

**FRAME**

Gérard Debreu, Theory of Value, page 49

``Three phenomena that the present analysis does not cover must be emphasized: (1) external economies and diseconomies, i.e., the case where the production set of a producer depends on the productions of the other producers (and/or on the consumptions of consumers), (2) increasing returns to scale, (3) the behavior of producers who do not consider prices as given in choosing their productions."

**FRAME**

Firm's behaviour with increasing returns

In the definition of a Walras equilibrium or competitive equilibrium, the producers are supposed to take the prices as given and to maximize the profit.

When the production exhibits increasing returns to scale, fixed cost or more general types of non-convexities, a competitive equilibrium may not exist since the supply may be empty or discontinuous.

Optimality and marginal cost

Hotelling, H., The Relation of Prices to Marginal Costs in an

Optimum System, {\it Econometrica}, 7 (1939), 151-155.

An optimum of welfare ``corresponds to the sale of everything at marginal cost."

When the production set is convex, marginal cost ``is equivalent" to profit maximization and the 2d Theorem of Welfare economics translates formally the claim of Hotelling.

The mathematical problem: how to define marginal cost in a general setting? What is the rule when the cost is not differentiable or the cost is not minimized at equilibrium?

**FRAME**

Marginal cost and normal cone: the smooth case

Marginal cost can be interpreted as a first order necessary condition for profit maximization. A production set $Y$ is a subset of the commodity space $L$ satisfying the free-disposal assumption $Y- L\_+=Y$.

**FRAME**

Marginal cost and normal cone: the non-smooth case

\begin {center}

$N^L\_Y(y)=N^C\_Y(y)$\hskip 2 cm $N^C\_Y(y)$\hskip 2.5 cm$N^L\_Y(y)$\hskip 3 cm

**FRAME**

Different normal cones used in economics

\begin{tabular}{llll}

Minkovski&&Debreu (1959)&Profit maxim\cr

&&&\cr\pause

Dubovickii-Miljutin&&Guesnerie (1975)&2d Welfare Thm\cr

&&&\cr\pause

Clarke&$N^C\_Y(y)$&Cornet (1990)&MP Equilibrium\cr

&&&\cr\pause

Mordukhovich&$N^L\_Y(y)$&Khan (1999) &2d Welfare Thm\cr

&&&\cr\pause

Cornet-Czarnecki&$N^I\_Y(y)$&BCC (2005)& MP Equilibrium\cr

&&&\cr\pause

Bonnisseau&${\mathcal{N}}\_Y(y)$& B (2002)& MP Equilibrium\cr

\end{tabular}

**FRAME**

The right concept for economic applications}

Two main problems:

- second welfare theorem

- existence of marginal (cost) pricing equilibrium

Two types of proofs:

- first order necessary conditions or ``non-convex" separation theorem

- fixed-point theorem.

Not a universal concept for both problems:

- $N^L$ for the second welfare theorem

- $N^L$ is not always compatible with the existence of an equilibrium (Beato and Mas-Colell's example).

**FRAME**

Definitions

{\textcolor{red}{The cone of perpendicular vectors}}

$$N\_Y^P(\bar y)=\{p \in \RR^\ell \mid \exists \rho >0, B(\bar y+\rho p, \rho \Vert p\Vert) \cap Y=\emptyset\}$$

$\exists \rho>0$ such that $\bar y$ maximizes $p\cdot y - \frac{1}{2 \rho} \Vert y - \bar y \Vert^2$ over $Y$.

{\textcolor{red}{ The limiting normal cone}} &\hskip 0.5 cm {\textcolor{red}{ Clarke's normal cone}} \cr

Mordukhovich & \hskip 0.5 cm Cornet\cr

$N\_Y^L(y)= \limsup\_{y^\prime \in Y, y^\prime \to y} N\_Y^P(y^\prime)$ & \hskip 0.5 cm$N\_Y^C(y)=\cl\co N\_Y^L(y)$\cr

\hskip 2 cm{\includegraphics[scale=0.1]{Fig\_coneLim.pdf}}& \hskip 2 cm{\includegraphics[scale=0.1]{Fig\_coneCla.pdf}}\cr

**FRAME**

The intermediate normal cone

$d\_Y$: the distance function to $Y$, is Lipschitzian, hence almost everywhere differentiable (Rademacher). $D(\nabla d\_Y)$ is the domain on which $d\_Y$ is differentiable. We let

$\partial d\_Y(y) =\bar\co \{p \in \RR^\ell \mid \exists (y^\nu) \subset D(\nabla d\_Y), y^\nu \to y, \nabla d\_Y(y^\nu ) \to p\}$

\begin{definition}

{\textcolor{red}{The intermediate normal cone}} is defined by:

$$N\_Y^I(y)=\cup\_{t \ge 0} t \limsup\_{y^\prime \notin Y, y^\prime \to y} \partial d\_Y(y^\prime)$$

\end{definition}

**FRAME**

A tighter marginal pricing rule

Cornet-Czarnecki

\begin{center}

for all $y \in \partial Y$, $MP(y)=N^I\_Y(y)$.

\end{center}

The interest of this definition is that it provides a set of prices, which can be smaller than Clarke's normal cone.

For the second welfare theorem, the limiting normal cone is already known as the smallest possible one.

**FRAME**

Marginal pricing rule in infinite dimensional spaces

Commodity spaces for the modeling of time and uncertainty, discrete or continuous time

- $L\_p^\ell (\Omega, \Sigma, \mu)$ or $L\_p^\ell ([0,1])$ $1 \le p \le +\infty$\\

pour la finance ou l’horizon infini ou l’incertain

-$M(K)$ of all finite countably additive signed measures on a compact metric space of characteristics, for location model or commodity differentiation\\

- Riesz spaces with a partial ordering to distinguish inputs from outputs.

A global framework which encompasses the previous cases

**FRAME**

An adapted definition

$L= {\mathcal{L}}^{\infty}(M,\cal M , \mu)$, $\Pi=ba(M,

\cal M , \mu)$

$S= \{ p\in\Pi\_+ \mid p({\chi})= 1 \}$, where $\chi$

is the function equal to $1$ for every $m$ in $M$.

Let $y \in Y$ and let $\rho>0$. The ${\mathcal{T}}^\rho\_Y(y)$

is the set of vector $v \in L$ which satisfies the following condition :

there exists $\eta >0$, for all $r>0$, there exists a weak$^\*$-open neighborhood $U$ of $y$

and $\varepsilon >0$, such that for all $y^\prime \in B(y,\rho)\cap U\cap Y$,

for all $t\in (0,\varepsilon)$,

$$[\{y^\prime\}+t B(v+\eta(y-y^\prime),r)]\cap Y\neq\emptyset$$

The small tangent cone ${\mathcal{T}}\_Y(y) = \cap\_{\rho>0} {\mathcal{T}}^\rho\_Y(y)$

and the large normal cone $p \in {\mathcal{N}}\_Y(y)$ if $p(v)\le 0$ for all $v \in {\mathcal{T}}\_Y(y)$.

The marginal pricing rule

MP(y) = {\mathcal{N}}\_{Y}(y)

\cap S

**FRAME**

Marginal pricing rule with external effects

Commodity space $R^\ell$

A production set $Y\_j$ is defined by a

correspondence $Y\_j$ from $R^{\ell(m+ n) }$ to $R^{\ell}$. For every $z \in R^{\ell( m +n) }$, $Y\_j(z)$ is the

production set given the environment $z$.

$MP\_j(y\_j,z)$ (Bonnisseau-M\'edecin) is the convex hull of

\[\left\{ p \in S \left|

\begin{array}{ll} \exists (z^\nu) \subset R^{\ell(m+n)}, & (z^\nu) \to z, \\

\exists (y\_{j}^\nu) \subset R^\ell, & (y\_{j}^\nu) \to y\_{j}, y\_{j}^\nu\in

\partial Y\_j(z^\nu), \\ \exists (p^\nu) \subset S, & (p^\nu) \to p, p^\nu \in

N\_{Y\_j(z^\nu)}(y\_j^\nu) \end{array}\right. \right\}.

\]

**FRAME**

Infinitely many commodities and external effects

$L$, the commodity space, a Banach lattice equipped with a lattice norm and a compatible Hausdorff locally convex-solid topology $\tau$. $e$ is a ``reference" commodity bundle, which is comparable with the total endowments of the economy.

\medskip

The production set $Y\_j$ is defined by a correspondence from $L^{(m+n)}$ to $L$.

**FRAME**

A mixed of the previous approaches

Bonnisseau-Fuentes

\[

\hat {\mathcal{T}}^{\rho}\_{Y\_{j}(z)}({\bar y\_{j}}):=\left\{

\nu\in{L}\left\vert

\begin{array}{c}

\exists \ {\eta}>0\ \textrm{such that}\ \forall\ {r}>0,\\

\exists V \in {\mathcal{V}}\_{\prod\_{L^{I+J}}\tau}(z) , \ \ U \in \\{\mathcal{V}}\_\tau(\bar y\_j) \ \varepsilon>0\ \mid\\

\forall\ \ {z'}\in B(z, \rho)\cap V,\\

\forall\ \ y^\prime\_j\in B(\bar y\_j, \rho)\cap U\cap Y\_j(z^{\prime}) \\

\textrm{and}\ \forall\ \ t\in (0,\varepsilon),\ \exists\ \xi\in r[-e, \ e]\ \textrm{such that}\\

\ y^\prime\_j+t(\nu+\eta(\bar y\_{j}-y^\prime\_j) + \xi) \in Y\_j(z^\prime)

\end{array}

\right.

\right\}

\]

\[

\hat{\mathcal{T}}\_{Y\_{j}({z})}({\bar y\_{j}}) := \cap\_{{\rho}>0}\hat{\mathcal{T}}^{\rho}\_{Y\_{j}({z})}({\bar y\_{j}}).

\]

\[

\hat{\mathcal{N}}\_{Y\_{j}({z})}({\bar y\_{j}})

=[\hat{\mathcal{T}}\_{Y\_{j}({z})}({\bar y\_{j}})]^o=\{{\pi}\in{L^\*}\mid{\pi({\nu})}\leq0\ \forall{\nu}\in\hat{\mathcal{T}}\_{Y\_{j}({z})}(\bar y\_{j})\}.

\]

**FRAME**

An internal consistency

Why this definition and not another one already existing in the mathematical literature, interaction between mathematics and economic interpretation

- Convex case;\\

- Finite dimensional case;\\

- Smooth case;\\

- Compatibility with the existence of an equilibrium under rather weak assumptions on the fundamentals of an economy;\\

We can extend the domain of validity for the existence of an equilibrium in such a way that non convex production sets are compatible with the existence of an equilibrium, but the optimality is no more guaranteed so the need for a public intervention and the importance of the role of income distribution.